THEORIES OF FAILURE

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- ✓ In case of material subjected to simple state of stress (tension or compression), failure occurs when the stress in the material reaches the elastic limit stress.
- ✓ In case of material subjected to complex stresses, the stage of failure is determined either to practically or theoretically.
- ✓ Non-applicability of any one theory to all states of stresses and to all materials has resulted in propagation of different theories relating the complex stresses to elastic limit in simple tension or compression.
- ✓ Since the complex stress system can be simplified into three principal stresses, the problem reduced to linking the three principal stresses to the stresses at elastic limit in case of simple stresses.

- The common most theories are
    1. Maximum principal stress theory
    2. Maximum Principal strain theory
    3. Maximum shear stress theory
    4. Maximum strain energy theory
    5. Maximum shear strain energy theory

1. Maximum Principal stress theory
or
Maximum normal stress theory

- This theory was proposed by Rankine.
- It states that failure will occur when the maximum principal stress (σ_1) in the complex system reaches the value of maximum stress (σ_{yt}) at the elastic limit simple tension or the minimum principal stress (i.e. maximum principal compression stress) reaches the elastic limit (σ_{yc}) in simple compression.

$$\sigma_1 = \sigma_{yt} \quad \text{in simple tension}$$

$$|\sigma_3| = \sigma_{yc} \quad \text{in simple compression}$$

- For the design, the maximum principal stress should not exceed the working stress σ for the material. $\sigma_1 \leq \sigma$,
- Working stress, $\sigma = \dfrac{\sigma_y}{F}$

  F : Factor of safety
- This theory is valid for brittle metals such as cast iron.
- Maximum principal stress theory is valid for thin walled tubes.

The maximum principal stress theory is contradicted in the following cases.

i. Failure in simple tension is caused by sliding at 45^0 with the axis of the specimen, there by failure occurred due to maximum shear stress and not due to direct tensile stress.

ii. The material which is weak in simple compression can sustain large hydrostatic pressure in excess of the elastic limit in simple compression.

Maximum principal strain theory

- This theory was proposed by Saint Venant.
- It states that the failure of a material occurs when the major principal tensile strain reaches the strain at the elastic limit in simple tension or when the minor principal strain (i.e maximum principal compressive strain) reaches the strain at elastic limit in simple compression.
- This theory is more appropriate for ductile materials, brittle materials and materials under hydrostatic pressure.
- It does not fit well with the experimental results.

Principal strain in the direction of principal stress σ_1, $e_1 = \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)]$

Principal strain in the direction of principal stress σ_3, $e_3 = \frac{1}{E}[\sigma_3 - \mu(\sigma_1 + \sigma_2)]$

According to maximum principal strain theory, the conditions to cause failure are

$$e_1 > \frac{\sigma_{yt}}{E} \;;\quad \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)] > \frac{\sigma_{yt}}{E} \Rightarrow \sigma_1 - \mu(\sigma_2 + \sigma_3) > \sigma_{yt} \qquad \text{or}$$

$$|e_3| > \frac{\sigma_{yc}}{E} \;;\quad \frac{1}{E}[\sigma_3 - \mu(\sigma_1 + \sigma_2)] > \frac{\sigma_{yc}}{E} \Rightarrow \sigma_3 - \mu(\sigma_1 + \sigma_2) > \sigma_{yc}$$

To prevent failure

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) < \sigma_{yt}$$

$$\sigma_3 - \mu(\sigma_1 + \sigma_2) < \sigma_{yc}$$

At the point of failure

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) = \sigma_{yt}$$

$$\left|\sigma_3 - \mu(\sigma_1 + \sigma_2)\right| = \sigma_{yc}$$

For the design purposes

$$\sigma_3 - \mu(\sigma_1 + \sigma_2) = \sigma_t$$

$$\sigma_3 - \mu(\sigma_1 + \sigma_2) = \sigma_c$$

Where σ_t and σ_c are the safe stresses.

Maximum shear stress theory

- This theory is also called Coulomb Guest's or Treasca's theory
- It states that the material will fail when the maximum shear stress ($\tau_{\max}$) in the complex system reaches the value of maximum shear stress in simple tension at the elastic limit.

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{yt}}{2}$$

$$\sigma_1 - \sigma_3 = \sigma_{yt}$$

$$\sigma_1 - \sigma_3 = \sigma_t$$

- This theory gives good correlation with the results of experiments on ductile materials.
- It gives satisfactory results for ductile materials particularly in case of shafts.
- The theory does not give accurate results for the state of stress of pure shear.
- The theory is not applicable in the case where the state of stress consists of triaxial tensile stresses of nearly equal magnitude.

4. Maximum strain energy theory

- This theory was proposed by Beltrani-Haigh.
- It states that the failure of a material occurs when the total strain energy in the material reaches the total strain energy of the material at the elastic limit in simple tension.

In a 3D stress system, the strain energy per unit volume is given by

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

At the point of failure

$$\frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] = \frac{\sigma_y^2}{2E}$$

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_y^2$$

In $2D$ stress system ($\sigma_2 = 0$), the above equation reduce to

$$\sigma_1^2 + \sigma_3^2 - 2\mu\sigma_1\sigma_3 = \sigma_y^2$$

For the design

$$\sigma_1^2 + \sigma_3^2 - 2\mu\sigma_1\sigma_3 \le \sigma^2$$

- It is applicable for ductile materials particularly in case of pressure vessel.
- The theory does not applicable to materials for which σ_{yt} is different from σ_{yc}
- The theory does not give results exactly equal to the experimental results even for ductile materials

If a body is subjected to high hydrostatic pressure equal compressive stress on all the three faces, then based on the strain energy theory, the maximum compressive stress will be

$$\sigma = \frac{\sigma_y}{\sqrt{3(1-\mu)}}$$

Yield stress of the material $= \sigma_y$

Strain energy stored, $U = \frac{1}{2}.\sigma_1.e_1 + \frac{1}{2}\sigma_2.e_2 + \frac{1}{2}.\sigma_3.e_3$

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma$$

$$e_1 = e_2 = e_3 = \frac{\sigma}{E} - 2\mu\frac{\sigma}{E} = \frac{\sigma}{E}(1-2\mu)$$

$$U = 3.\sigma.\frac{\sigma}{E}(1-2\mu) = \frac{3\sigma^2}{E}(1-2\mu)$$

Also $U =$ stress $\times$ strain

$$= \sigma_y.\frac{\sigma_y}{E} = \frac{\sigma_y^2}{E}$$

$$\frac{3\sigma^2}{E}(1-2\mu) = \frac{\sigma_y^2}{E} \Rightarrow \quad \sigma = \frac{\sigma_y}{\sqrt{3(1-\mu)}}$$

5. Maximum shear strain energy theory
or
Distortion energy theory

- This theory was proposed by Von Mises-Henky
- It states that the elastic failure occurs when the shear strain energy per unit volume in the stressed material reaches a value equal to the shear strain energy per unit volume at the elastic limit point in tension.
- The theory gives best results for ductile material particularly in case of pure shear or $\sigma_{yc} = \sigma_{yt}$.

Shear strain energy per unit volume due to principal stresses σ_1, σ_2 and σ_3,

$$U_S = \frac{1+\mu}{3E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_3\sigma_2 + \sigma_1\sigma_3)\right]$$

$$U_S = \frac{1+\mu}{6E}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

$$U_S = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

For the simple tension at the elastic limit point, $(\sigma_1 = \sigma_{yt}, \sigma_2 = \sigma_3 = 0)$, the shear strain energy per unit volume is given by

$$U'_S = \frac{1}{12G}\left[(\sigma_{yt} - 0)^2 + (0-0)^2 + (0-\sigma_{yt})^2\right] = \frac{1}{12G}.2\sigma_{yt}^2$$

Equating the two strain energies, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2.\sigma_t^2$

In $2D$ stress system ($\sigma_2 = 0$), the above equation reduce to

$$\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3 = \sigma_t^2$$

Design conditions for various failure theory

Failure theory	Proposed by	Condition for design
Maximum principal stress theory	Rankine, Lame	$\sigma_1 \leq \sigma$
Maximum principal strain theory	Saint Venants	$\frac{1}{E}(\sigma_1 - \mu\sigma_2) \leq \frac{\sigma_y}{E}$
Maximum shear stress theory	Coulomb Guest, Tresca	$\frac{\sigma_1 - \sigma_2}{2} \leq \frac{\sigma_y}{2}$
Maximum strain energy theory	Beltrami-Haigh	$(\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2) \leq \sigma_y^2$
Distortion energy theory	Huber-Henky-Von Mises	$(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2) \leq \sigma_y^2$

- When one of the principal stresses at a point is large in comparison to the other, all the failure theories gives nearly the same result.
- When a member is subjected to uni-axial tension, all the failure theories gives the same result.

GATE
PREVIOUS QUESTIONS
AND
SOLUTIONS

01. A small element at the critical section of a component is in a bi-axial state of stress with the two principle stresses being 360 MPa and 140 MPa. The maximum working stress according to distortion energy theory is: GATE ME 1997

a. 220 MPa   b. 110 MPa   c. 314 MPa   d. 330 MPa

01. C.

σ_1 : Major principal stress = 360 MPa

σ_2 : Minor principal stress = 140 MPa

f : working stress in the element according to distortion energy theory

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 \geq f^2$$

$$360^2 + 140^2 - 360 \times 140 \geq f^2$$

$$f = 314.3 \text{ N/mm}^2$$

02. According to Von-Mises distortion energy theory, the distortion energy under three dimensional stress state is represented by GATE ME 2006

a. $\frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\mu(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

b. $\frac{1-2\mu}{6E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+2(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

c. $\frac{1+\mu}{3E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

d. $\frac{1}{3E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-\mu(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

04. The homogenous state of stress for a metal part undergoing plastic deformation is

$$T = \begin{pmatrix} 10 & 5 & 0 \\ 5 & 20 & 0 \\ 0 & 0 & -10 \end{pmatrix}$$

Where the stress component values are in MPa. Using Von-Mises yield criterion, the value of estimated shear yield stress, in MPa is GATE ME 2012

a. 9.50 b. 16.07 c. 28.52 d. 49.41

04. B

04. B

$$\text{State of Stress, } T = \begin{bmatrix} 10 & 5 & 0 \\ 5 & 20 & 0 \\ 0 & 0 & -10 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

$\sigma_x = 10$ MPa, $\sigma_y = 20$ MPa, $\sigma_z = -10$MPa

$\tau_{xy} = 5$MPa, $\tau_{yz} = 0$, $\tau_{zx} = 0$

$$\text{Principal stresses, } \sigma_{1,3} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4.\tau_{xy}^2}$$

$$\sigma_{1,3} = \frac{10+20}{2} \pm \frac{1}{2}\sqrt{(10-20)^2 + 4\times 5^2} = 15 \pm \frac{1}{2}\sqrt{100+100} = 15 \pm 7.07$$

$\sigma_1 = 15 + 7.07 = 22.07$ MPa

$\sigma_2 = 15 - 7.07 = 7.93$ MPa

σ_{yt} : Yield stress in tension

σ_{ys} : Yield stress in shear

According to Von-Mises yield criterion,

$$(\sigma_{yt})^2 \geq \frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

$$(\sigma_{yt})^2 \geq \frac{1}{2}\left[(22.07 - 7.93)^2 + (7.93 + 10)^2 + (-10 - 22.07)^2\right]$$

$$(\sigma_{yt})^2 \geq 774.9 \Rightarrow \sigma_{yt} \geq 27.84\,\text{MPa}$$

$$\sigma_{ys} = \frac{\sigma_{yt}}{\sqrt{3}} = \frac{27.84}{\sqrt{3}} = 16.07\,\text{MPa}$$

05. A machine element is subjected to the following bi-axial state of stress: $\sigma_x = 80$; $\sigma_y = 20$; $\tau_{xy} = 40$. If the shear strength of the material is 100 MPa, the factor of safety as per Tresca's maximum shear stress theory is GATE ME 2015

a. 1.0 **b**. 2.0 c. 2.5 d. 3.3

05. B

Biaxial state of stress for an element:

$\sigma_x = 80\,\text{MPa}$, $\sigma_y = 20\,\text{MPa}$ and $\tau_{xy} = 40\,\text{MPa}$

Shear strength of the material, $\tau = 100\,\text{MPa}$

Maximum shear stress induced in an element, $\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

$$= \frac{1}{2}\sqrt{(80-20)^2 + 4(40)^2} = 50\ \text{MPa}$$

Factor of safety, $F = \dfrac{\text{shear strength}}{\text{shear stress induced}} = \dfrac{100}{50} = 2$

06. The uniaxial yield stress of a materials is 300 MPa. According to von Mises criterion, the shear yield stress (in MPa) of the material is GATE ME 2015

Ans: 171 to 175

06. 173.2

Uniaxial yield stress of material, $\sigma_{yt} = 300\,\text{MPa}$

Shear yield stress, $\sigma_{ys} = \dfrac{\sigma_{yt}}{\sqrt{3}} = \dfrac{300}{\sqrt{3}} = 173.2$ MPa

07. The principal stresses t a point in a critical section of a machine component are $\sigma_1 = 60$ MPa, $\sigma_2 = 5$ MPa and $\sigma_3 = -40$ MPa. For the material of the component, the tensile yield strength is $\sigma_y = 200\,\text{MPa}$. According to the maximum shear stress theory, the factor of safety is GATE ME 2017

a.1.67 **b.** 2.00 b. 3.60 d. 4.00

07. B

Principal stresses at a point in machine component

$\sigma_1 = 60\,\text{MPa}$, $\sigma_2 = 5\,\text{MPa}$ and $\sigma_3 = -40\,\text{MPa}$

Tensile yield strength, $\sigma_{yt} = 200\,\text{MPa}$

Permissible shear stress, $\tau = \dfrac{\sigma_{yt}}{2} = \dfrac{200}{2} = 100$ MPa

Maximum shear stress, $\tau_{\max} = \dfrac{\sigma_1 - \sigma_3}{2}$

$$= \frac{60-(-40)}{2} = 50 \text{ MPa}$$

Factor of safety, $F = \dfrac{\text{Permissible shear stress}}{\text{Maximum shear stress induced}} = \dfrac{100}{5} = 2$

42. At a critical point in a component, the state of stress is given as $\sigma_{xx} = 100 MPa$, $\sigma_{xy} = 220 MPa$, $\sigma_{xy} = \sigma_{yx} = 80 MPa$ and all other stress components are zero. The yield strength of the material is 468 MPa. The factor of safety on the basis of maximum shear stress theory is…. (round off to one decimal place).

GATE ME 2019

42. 1.8

State of stress at a critical point:

σ_x =100 MPa, σ_y =220MPa and τ =80 MPa

Yield strength of the material, σ_y =468 MPa

Factor of safety, $F = ?$

Principal stresses are, $\sigma_{1,2} = \dfrac{\sigma_x + \sigma_y}{2} \pm \dfrac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

$$\sigma_{1,2} = \frac{100+220}{2} \pm \frac{1}{2}\sqrt{(100-220)^2 + 4\times 80^2}$$

$$= 160 \pm 100$$

σ_1 =160+100=260MPa

σ_2 =160-100=60MPa

Maximum shear stress, $\tau_{\max} = \dfrac{\sigma_1 - \sigma_2}{2} = \dfrac{260-60}{2}$ =100MPa

According to maximum shear stress theory,

$$\tau_{\max} = \frac{\sigma_y}{2F} = \max\left\{\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_1}{2} \text{ and } \frac{\sigma_2}{2}\right\}$$

$$\frac{468}{2\times F} = \frac{260}{2} \Rightarrow F = 1.8$$